

Partial Differential Equations

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Partial Differential Equations

- Definition
- One of the classical partial differential equation of mathematical physics is the equation describing the conduction of heat in a solid body (Originated in the 18th century). And a modern one is the space vehicle reentry problem: Analysis of transfer and dissipation of heat generated by the friction with earth's atmosphere.

For example:

- Consider a straight bar with uniform cross-section and homogeneous material. We wish to develop a model for heat flow through the bar.
- Let $u(x,t)$ be the temperature on a cross section located at x and at time t . We shall follow some basic principles of physics:
- **A.** The amount of heat per unit time flowing through a unit of cross-sectional area is proportional to $\partial u / \partial x$ with constant of proportionality $k(x)$ called the **thermal conductivity** of the material.

- **B.** Heat flow is always from points of higher temperature to points of lower temperature.
- **C.** The amount of heat necessary to raise the temperature of an object of mass “m” by an amount Δu is a “**c(x) m Δu ”**, where **c(x)** is known as the **specific heat** capacity of the material.
- Thus to study the amount of heat $H(x)$ flowing from left to right through a surface A of a cross section during the time interval Δt can then be given by the formula:

$$H(x) = -k(x)(\text{area of } A)\Delta t \frac{\partial u}{\partial x}(x, t)$$

Likewise, at the point $x + \Delta x$,
we have

- Heat flowing from left to right across the plane during an time interval Δt is:

$$H(x + \Delta x) = -k(x + \Delta x)(\text{area of B})\Delta t \frac{\partial u}{\partial t}(x + \Delta x, t).$$

- If on the interval $[x, x + \Delta x]$, during time Δt , additional heat sources were generated by, say, chemical reactions, heater, or electric currents, with energy density $Q(x, t)$, then the total change in the heat ΔE is given by the formula:

$\Delta E = \text{Heat entering A} - \text{Heat leaving B} +$
 $\text{Heat generated} .$

- And taking into simplification the principle C above, $\Delta E = c(x) m \Delta u$, where $m = \rho(x) \Delta V$.
After dividing by $(\Delta x)(\Delta t)$, and taking the limits as Δx , and $\Delta t \rightarrow 0$, we get:

$$\frac{\partial}{\partial x} \left[k(x) \frac{\partial u}{\partial x} (x, t) \right] + Q(x, t) = c(x) \rho (x) \frac{\partial u}{\partial t} (x, t)$$

- If we assume k , c , ρ are constants, then the eq.

Becomes:

$$\frac{\partial u}{\partial t} = \beta^2 \frac{\partial^2 u}{\partial x^2} + p(x, t)$$

Boundary and Initial conditions

- Remark on boundary conditions and initial condition on $u(x,t)$.
- We thus obtain the mathematical model for the heat flow in a uniform rod without internal sources ($p = 0$) with homogeneous boundary conditions and initial temperature distribution $f(x)$, the following Initial Boundary Value Problem:

One Dimensional Heat Equation

$$\frac{\partial u}{\partial t}(x, t) = \beta \frac{\partial^2 u}{\partial x^2}(x, t), \quad 0 < x < L, \quad t > 0,$$

$$u(0, t) = u(L, t) = 0, \quad t > 0,$$

$$u(x, 0) = f(x), \quad 0 < x < L.$$

The method of separation of variables

- Introducing solution of the form
- $u(x,t) = X(x) T(t)$.
- Substituting into the I.V.P, we obtain:

$$X(x)T'(t) = \beta X''(x)T(t), \quad 0 < x < L, \quad t > 0.$$

this leads to the following eq.

$$\frac{T'(t)}{\beta T(t)} = \frac{X''(x)}{X(x)} = \text{Constants.} \quad \text{Thus we have}$$

$$T'(t) - \beta kT(t) = 0 \quad \text{and} \quad X''(x) - kX(x) = 0.$$

Boundary Conditions

- Imply that we are looking for a non-trivial solution $X(x)$, satisfying:

$$X''(x) - kX(x) = 0$$

$$X(0) = X(L) = 0$$

- We shall consider 3 cases:
- $k = 0$, $k > 0$ and $k < 0$.

- Case (i): $k = 0$. In this case we have
 - $X(x) = 0$, trivial solution
- Case (ii): $k > 0$. Let $k = \lambda^2$, then the D.E gives $X'' - \lambda^2 X = 0$. The fundamental solution set is: $\{ e^{\lambda x}, e^{-\lambda x} \}$. A general solution is given by:

$$X(x) = c_1 e^{\lambda x} + c_2 e^{-\lambda x}$$
 - $X(0) = 0 \Rightarrow c_1 + c_2 = 0$, and
 - $X(L) = 0 \Rightarrow c_1 e^{\lambda L} + c_2 e^{-\lambda L} = 0$, hence
 - $c_1 (e^{2\lambda L} - 1) = 0 \Rightarrow c_1 = 0$ and so is $c_2 = 0$.
 - Again we have trivial solution $X(x) \equiv 0$.

Finally Case (iii) when $k < 0$.

- We again let $k = -\lambda^2$, $\lambda > 0$. The D.E. becomes:
- $X''(x) + \lambda^2 X(x) = 0$, the auxiliary equation is
- $r^2 + \lambda^2 = 0$, or $r = \pm \lambda i$. The general solution:
- $X(x) = c_1 e^{i\lambda x} + c_2 e^{-i\lambda x}$ or we prefer to write:
- $X(x) = c_1 \cos \lambda x + c_2 \sin \lambda x$. Now the boundary conditions $X(0) = X(L) = 0$ imply:
- $c_1 = 0$ and $c_2 \sin \lambda L = 0$, for this to happen, we need $\lambda L = n\pi$, i.e. $\lambda = n\pi / L$ or $k = - (n\pi / L)^2$.
- We set $X_n(x) = a_n \sin (n\pi / L)x$, $n = 1, 2, 3, \dots$

Finally for $T'(t) - \beta k T(t) = 0$, $k = -\lambda^2$.

- We rewrite it as: $T' + \beta \lambda^2 T = 0$. Or $T' = -\beta \lambda^2 T$. We see the solutions are

condition $u \in \pi \lambda$:

the boundary conditions. To satisfy the initial

$u^n(x, t) = X^n(x) T^n(t)$ satisfies the D.E and

Thus the function

$$T^n(t) = \rho^n e^{-\beta (\lambda^n / \Gamma) t}, \quad n = 1, 2, 3, \dots$$

$$u(x,t) = \sum u_n(x,t), \text{ over all } n.$$

- More precisely,

$$u(x,t) = \sum_1^{\infty} c_n e^{-\beta \left(\frac{n\pi}{L}\right)^2 t} \sin\left(\frac{n\pi}{L}\right)x.$$

We must have :

$$u(x,0) = \sum_1^{\infty} c_n \sin\left(\frac{n\pi}{L}\right)x = f(x).$$

- This leads to the question of when it is possible to represent $f(x)$ by the so called
- **Fourier sine series ??**

Jean Baptiste Joseph Fourier (1768 - 1830)

- Developed the equation for heat transmission and obtained solution under various boundary conditions (1800 - 1811).
- Under Napoleon he went to Egypt as a soldier and worked with G. Monge as a cultural attache for the French army.

Example

- Solve the following heat flow problem

$$\frac{\partial u}{\partial t} = 7 \frac{\partial^2 u}{\partial x^2} , \quad 0 < x < \pi , \quad t > 0.$$

$$u(0, t) = u(\pi, t) = 0 , \quad t > 0 ,$$

$$u(x, 0) = 3 \sin 2x - 6 \sin 5x , \quad 0 < x < \pi.$$

- Write $3 \sin 2x - 6 \sin 5x = \sum c_n \sin (n\pi/L)x$, and comparing the coefficients, we see that $c_2 = 3$, $c_5 = -6$, and $c_n = 0$ for all other n . And we have $u(x, t) = u_2(x, t) + u_5(x, t)$.

Wave Equation

- In the study of vibrating string such as piano wire or guitar string.

$$\frac{\partial^2 u}{\partial t^2} = \alpha^2 \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < L, \quad t > 0,$$

$$u(0, t) = u(L, t), \quad t > 0,$$

$$u(x, 0) = f(x), \quad 0 < x < L,$$

$$\frac{\partial u}{\partial t}(x, 0) = g(x), \quad 0 < x < L.$$

Example:

- $f(x) = 6 \sin 2x + 9 \sin 7x - \sin 10x$, and
- $g(x) = 11 \sin 9x - 14 \sin 15x$.
- The solution is of the form:

$$u(x, t) = \sum_{n=1}^{\infty} \left[a_n \cos \frac{n\pi\alpha}{L} t + b_n \sin \frac{n\pi\alpha}{L} t \right] \sin \frac{n\pi x}{L}.$$

Reminder:

- TA's Review session
- Date: July 17 (Tuesday, for all students)
- Time: 10 - 11:40 am
- Room: 304 BH

Final Exam

- Date: July 19 (Thursday)
- Time: 10:30 - 12:30 pm
- Room: LC-C3
- Covers: all materials
- I will have a review session on Wednesday

Fourier Series

- For a piecewise continuous function f on $[-T, T]$, we have the Fourier series for f :

$$f(x) \approx \frac{a_0}{2} + \sum_{n=1}^{\infty} \left\{ a_n \cos\left(\frac{n\pi}{T}x\right) + b_n \sin\left(\frac{n\pi}{T}x\right) \right\},$$

where

$$a_0 = \frac{1}{T} \int_{-T}^T f(x) dx, \text{ and}$$

$$a_n = \frac{1}{T} \int_{-T}^T f(x) \cos\left(\frac{n\pi}{T}x\right) dx; \quad n = 1, 2, 3, \dots$$

$$b_n = \frac{1}{T} \int_{-T}^T f(x) \sin\left(\frac{n\pi}{T}x\right) dx; \quad n = 1, 2, 3, \dots$$

Examples

- Compute the Fourier series for

$$f(x) = \begin{cases} 0, & -\pi < x < 0, \\ x, & 0 < x < \pi. \end{cases}$$

$$g(x) = |x|, \quad -1 < x < 1.$$

Convergence of Fourier Series

- Pointwise Convergence
- **Theorem.** If f and f' are piecewise continuous on $[-T, T]$, then for any x in $(-T, T)$, we have

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} \left\{ a_n \cos \frac{n\pi x}{T} + b_n \sin \frac{n\pi x}{T} \right\} = \frac{1}{2} \{ f(x^+) + f(x^-) \},$$

- where the a_n 's and b_n 's are given by the previous formulas. It converges to the average value of the left and right hand limits of $f(x)$. Remark on $x = T$, or $-T$.

Fourier Sine and Cosine series

- Consider Even and Odd extensions;
- Definition: Let $f(x)$ be piecewise continuous on the interval $[0, T]$. The Fourier cosine series of $f(x)$ on $[0, T]$ is:

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{T}\right)x, \quad a_n = \frac{2}{T} \int_0^T f(x) \cos\left(\frac{n\pi}{T}\right)x dx$$

- and the Fourier sine series is:

$$\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{T}\right)x, \quad b_n = \frac{2}{T} \int_0^T f(x) \sin\left(\frac{n\pi}{T}\right)x dx,$$

$$n = 1, 2, 3, \dots$$

Consider the heat flow problem:

$$(1) \quad \frac{\partial u}{\partial t} = 2 \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < \pi \quad t > 0,$$

$$(2) \quad u(0, t) = u(\pi, t), \quad t > 0,$$

$$(3) \quad u(x, 0) = \begin{cases} x, & 0 < x \leq \frac{\pi}{2}, \\ \pi - x, & \frac{\pi}{2} \leq x < \pi \end{cases}$$

Solution

- Since the boundary condition forces us to consider sine waves, we shall expand $f(x)$ into its Fourier Sine Series with $T = \pi$.

Thus

$$b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx dx$$

With the solution

$$u(x, t) = \sum_{n=1}^{\infty} b_n e^{-2n^2 t} \sin nx$$

where

$$b_n = \begin{cases} 0, & \text{if } n \text{ is even,} \\ \frac{4(-1)^{(n-1)/2}}{n^2 \pi} & \text{when } n \text{ is odd.} \end{cases}$$