Descarte's rule of signs:

a) If $f(x)$ be a polynomial with real coefficients.
The no. of positive roots of the equation $f(x) = 0$
is either equal to the no. of changes of sign in the successive coefficients of $f(x)$
or less than that of changes of sign by an even number.

i.e. if v *be* the no. of changes of sign in the successive coefficients of $f(x)$ &
if r *be the no. of* $(+)$*ve roots then* $v - r = $ *even positive number*

b) The no. of negative roots of the equation $f(x) = 0$
is either equal to the no. of changes of sign in the successive coefficients of $f(-x)$
or less than that of changes of sign by an even number

Ex: Apply Descarte's rule of signs to determine the nature of roots of $x^4 + x^2 + x - 1 = 0$

Ans: Let $f(x) = x^4 + x^2 + x - 1$

Sign of coeff. of	x^4	x^2	x	constant	No. of changes in sign
$f(x)$	$+$	$+$	$+$	$-$	1
$f(-x)$	$+$	$+$	$-$	$-$	1

By Descarte's rule of signs one positive root, one negative root. Since the equation is of degree four therefore it has four roots. Hence no. of imaginary root is 4 – 2 = 2

Ex: Apply Descarte's rule of signs to find the nature of the roots of $x^8 + 1 = 0$

$f(x) = x^8 + 1 \because$ no change in sign in coeff. of $f(x)$ $\therefore$ no $(+)ve$ roots.

Similarly there is no change in sign in coeff. of $f(-x)$ $\therefore$ no $(-)ve$ roots.

Therefore all the roots are imaginary.

Ex: Apply Descarte's rule of signs to find the nature of the roots of $x^3 + 1 = 0$

$f(x) = x^3 + 1 \because$ no change in sign in coeff. of $f(x)$ $\therefore$ no $(+)ve$ roots.

Similarly there is one change in sign in coeff. of $f(-x)$ $\therefore$ one $(-)ve$ root.

Therefore all other roots are imaginary.

From Descarte's rule of signs we cannot obtain exact no. of real roots of a polynomial equation $f(x) = 0$. But by strum's theorem we can find exact no. of real roots

Def: Sturm's functions

Let $f(x)$ be a polynomial of x of degree n. Let $f'(x)$ be the first derivative.

Divide $f(x)$ by $f'(x)$ and let $-f_2(x)$ be the remainder

Now divide $f'(x)$ by $f_2(x)$ and let the remainder be $-f_3(x)$.
Continue this process till we get the last remainder.

The functions $f(x), f'(x), f_2(x), f_3(x)$, --------, $f_n(x)$ are called sturm's function.

If $f(x) = 0$ has no equal roots, then last Sturm's function $f_n(x)$ is constant.
$\therefore$ we have $(n+1)$ Sturm's function.

Sturm's theorem: (All roots are unequal)

Let $f(x)$ be a polynomial in x of degree n. Let a, b be any two real nos $(a < b)$.
then the no. of distinct real roots of $f(x) = 0$ lying between a & b
is equal to the difference between no. of changes of signs when x is put equal to a
and the no. of changes of signs when x is put equal to b in the $(n+1)$

Sturm's functions

Conditions that all the roots are real & distinct:

For a polynomial $f(x)$ of degree n with leading coefficient positive there are $n+1$ Sturm′s functions

& the leading coefficients of all these functions must be positive

Ex: Use Sturm's functions to show the roots of the equation $x^3+3x^2-3=0$ are real & distinct.

$f(x)=x^3+3x^2-3 \;\therefore f'(x)=3x^2+6x$ take $f_1(x)=x^2+2x$

dividing $f(x)$ by $f_1(x)$ remainder is $-2x-3 \;\therefore f_2(x)=2x+3$

again dividing $f_1(x)$ by $f_2(x)$ remainder is $-3 \;\therefore f_3(x)=3$

$\therefore$ Sturm′s functions are

$f(x)=x^3+3x^2-3, f_1(x)=x^2+2x, f_2(x)=2x+3$ & $f_3(x)=3$

$f(x)$ is a polynomial of x of degree 3

there are 4 Sturm′s functions & leading coefficients are positive

$\therefore$ all the roots are real & distinct.

Ex: Calculate Sturm's functions and locate the position of the real roots of $x^3 - 3x - 1 = 0$

Let $f(x) = x^3 - 3x - 1$ $f'(x) = 3x^2 - 3$ $take\ f_1(x) = x^2 - 1$

dividing $f(x)$ by $f_1(x)$ remainder is $-2x - 1$ $\therefore f_2(x) = 2x + 1$

again dividing $f_1(x) by\ f_2(x)$ remainder is -3 $\therefore f_3(x) = 3$

$\therefore$ Sturm's functions are

$f(x) = x^3 - 3x - 1, f_1(x) = x^2 - 1, f_2(x) = 2x + 1\ \&\ f_3(x) = 5$

x \ sign of	$f(x) = x^3 - 3x - 1$	$f_1(x) = x^2 - 1$	$f_2(x) = 2x + 1$	$f_3(x) = 5$	No. of changes of sign
$-\infty$	$-$	$+$	$-$	$+$	3
0	$-$	$-$	$+$	$+$	1
∞	$+$	$+$	$+$	$+$	0
-2	-	+	-	+	3
-1	+	0	-	+	2
0	-	-	+	+	1
1	-	0	+	+	1
2	+	+	+	+	0

For $x = -\infty$ $no. of\ changes\ in\ sign\ of\ Sturm's\ functions\ are$ 3

For $x = 0$ $no. of\ changes\ in\ sign\ of\ Sturm's\ functions\ is$ 1

$$\therefore\ number\ of negative\ roots = 3 - 1 = 2.$$

For $x = \infty$ $no. of\ changes\ in\ sign\ of\ Sturm's\ functions\ is$ 0

$$\therefore\ number\ of positive\ roots = 1 - 0 = 1$$

For $x = -2$ $no. of\ changes\ in\ sign\ of\ Sturm's\ functions\ is$ 3

For $x = -1$ $no. of\ changes\ in\ sign\ of\ Sturm's\ functions\ is$ 2

$$\therefore\ one\ root\ in\ (-2, -1)$$

For $x = 0$ $no. of\ changes\ in\ sign\ of\ Sturm's\ functions\ is$ 1

$$\therefore\ one\ root\ in\ (-1, 0)$$

For $x = 1$ $no. of\ changes\ in\ sign\ of\ Sturm's\ functions\ is$ 1

For $x = 2$ $no. of\ changes\ in\ sign\ of\ Sturm's\ functions\ is$ 0

$$\therefore\ one\ root\ in\ (1, 2)$$