BNC Mathematics Hons. Sem - V

- Let S be the event space of the random experiment E. If E is repeated n times then the event space of the compound experiment is $S^n = S \times S \times --- \times S$
- Bernoulli trials: S consists of two points 'success' & 'failure' & probability of 'success' or 'failure' in each trial is constant.
- Binomial law: $If\ A_i\ denotes\ the\ event\ 'exactly\ i\ successes'\ then$ $P(A_i) = \binom{n}{i} p^i q^{n-i}\ for\ i = 0,1,2,----,n$
- The most probable no. of successes in Bernoullian sequence of n trials is the

integers $i_m\ given\ by\ (n+1)p - 1 \leq i_m \leq (n+1)p$

- Poisson trials: S consists of two points 'success' & 'failure' & probability of 'success' or 'failure' in each trial is not constant.
- Poisson approximation to Binomial law: $\lim_{n \to \infty} \binom{n}{i} p^i q^{n-i} = e^{-\mu} \frac{\mu^i}{i!}$, where $\mu = \frac{p}{n}$

- Random variable: $X : S \to R$
- X is discrete or continuous according range of X is discrete or continuous.
- Event $X = a \equiv \{U \mid X(U) = a\} \therefore P(X = a) = P\{U \mid X(U) = a\}$
- X is one dimensional R.V. taking values $x_1, x_2, --- then\ p_i = Prob(X =$

D1. Let E be a random experiment with event space S having m event points. if we repeat the random experiment under identical conditions n times then we have n independent trials of E & is denoted by E_n and the corresponding event space is denoted by $\mathrm{S}^{\mathrm{n}} = S \times \mathrm{S} \times --- \times S$,

$i.e. certesian\ product\ of\ S$ n times, containg m^n event points.

where each event point is an n − tuple.

D2. Bernoulli trials: If the random experiment be such that its event space consists of only two event points 'success' & 'failure', then a sequence of independent trials of the experiment will be called a Bernoullian sequence of trials provided the probability of 'success' or 'failure' in each trial is constant.

Ex: if an unbiased coin is tossed n times under identical conditions then we have Bernoulli trials where p is the probability of head i.e. success in each trial is ½ & probability of tail i.e. failure is ½.

Binomial law: $If\ A_i\ denotes\ the\ event\ 'exactly\ i\ successes'\ (i \leq n)$

$in\ Bernoulli\ trials\ of\ the\ random\ experiment\ E.$

$then\ P(A_i) = \binom{n}{i}p^i q^{n-i}\ for\ i = 0,1,2,----,n$

$where\ p\ is\ the\ probability\ of\ success\ in\ each\ trial\ \&\ q = 1 - p$

Ex: A die is thrown 10 times in succession. Find the probability of obtaining six at least once.

Soln: Here n=10, p=probability of six in a throw=1/6 & A be the event at least one six in 10 throws.

Then $\bar{A}$ is the event no six in 10 throws. $\therefore P(\bar{A}) = \binom{10}{0}p^0 q^{10} = (\frac{5}{6})^{10} \therefore P(A) = 1 - (\frac{5}{6})^{10}$

Theorem: the most probable no. of successes in Bernoullian sequence of n trials is the

integers $i_m\ given\ by\ (n+1)p - 1 \leq i_m \leq (n+1)p$ where p is the constant probability of success in each trial

- Poisson trials: A sequence of independent trials of a random experiment E the event space of which contains two points success & failure, is called a poisson sequence of trials if the probability of success in each trial is not constant but varies from one trial to another.

- Ex: - consider three trials & the probabilities of success are p_1, p_2, p_3 .

$the\ event\ space\ is\ \{s,f\} \times \{s,f\} \times \{s,f\}$

$= \{(s,s,s),(s,s,f),(s,f,s),(f,s,s),(f,f,s),(f,s,f),(s,f,f),(f,f,f)\}$

Let event $A_i = i\ success \therefore A_3 = \{(s,s,s)\}, A_2 = \{(s,s,f),(s,f,s),(f,s,s)\}$

$A_1 = \{(f,f,s),(f,s,f),(s,f,f)\}, A_0 = \{(f,f,f)\}$

$\therefore P(A_0) = q_1q_2q_3\ P(A_1) = q_1q_2p_3 + q_1p_2q_3 + p_1q_2q_3,$

$P(A_2) = p_1p_2q_3 + p_1q_2p_3 + q_1p_2p_3$ & $P(A_3) = p_1p_2p_3$

Poisson approximation to Binomial law:

If $p = \frac{\mu}{n}, where\ \mu\ is\ a\ positive\ constant, 0 < p < 1, n\ is\ a\ positive\ integer, then$

$$\lim_{n\to\infty} \binom{n}{i} p^i q^{n-i} = e^{-\mu} \frac{\mu^i}{i!}$$

Probability distribution:

D1. If corresponding to every point U of an event space S we have by a given rule, a unique real value of X = X(U), i.e. X is a real valued function defined on S. then X is called a random variable. i.e. $X: S \to R$. The range of function X will be called the spectrum of the random variable.

The spectrum may be discrete or continuous and accordingly the random variable is said to be discrete or continuous.

R1. If X_1 & X_2 are random variables then

(1) X_1X_2, $C_1X_1 + C_2X_2$, $\max\{X_1, X_2\}$, $\min\{X_1, X_2\}$ are also random variables

(2) If X is a random variable & f is a continuous or increasing function then $f(X)$ is also a R.V.

(3) If $\{X_n\}$ be a sequence of random variable then $\sup X_n$, $Inf\ X_n$, $Lim\ Sup\ X_n$, $Lim\ Inf\ X_n$ all are R.V.

D2. By the event $X = a$ we mean the set of all event points U for which $X(U) = a$

$i.e.\ X = a \equiv \{U \mid X(U) = a\}\ \therefore P(X = a) = P\{U \mid X(U) = a\}$

D3. Let X be an one − dimensional discrete $R.V.$

taking at most a countably infinite no. of values $x_1, x_2, ---$

If with each possible outcomes we associate a no. $p_i = Prob(X = x_i) = p(x_i)$

called the probability of x_i such that $p(x_i) \geq 0\ \&\ \sum_i p(x_i) = 1$

then the function p is called the probability mass function (p. m. f) of the $R.V.X$

& the set $\{p(x_i)\}$ is called the probability distribution of the Random variable X

D4. The distribution function of a random variable X is a function of real variable x to be denoted by $F_X(x)$ *or* $F(x)$ defined in $(-\infty, \infty)$by

$$F(x) = P\,(-\infty < X \leq x)$$

Properties of distribution function:

P1. If $a < b$ & *if* $F(x)$ be the distribution function of the random variable X then $P(a < X \leq b) = F(b) - F(a)$

Events $(-\infty < X \leq a)$ & $(a < X \leq b)$ *are mutually exclusive* &

$$(-\infty < X \leq a) + (a < X \leq b) = (-\infty < X \leq b)$$

$$\therefore P\{(-\infty < X \leq a) + (a < X \leq b)\} = P(-\infty < X \leq b)$$

$$\Longrightarrow P(-\infty < X \leq a) + P\,(a < X \leq b) = P(-\infty < X \leq b)$$

$$\therefore F(a) + P(a < X \leq b) = F(b) \;\; i.e.\, P(a < X \leq b) = F(b) - F(a)$$

$F(b) - F(a) = P(a < X \leq b) \geq 0$

$\therefore F(b) \geq F(a)$ if $b > a$ $\therefore$ $F(x)$ is a monotonic non decreasing function of x

$P(a \leq X \leq b) = F(b) - F(a) + P(X = a)$

$P(a < X < b) = F(b) - F(a) - P(X = b)$

$P(a \leq X < b) = F(b) - F(a) - P(X = b) + P(X = a)$

P2. $F(-\infty) = 0, F(\infty) = 1, F(a) - F(a - 0) = P(X = a), F(a + 0) = F(a)$

Let $A_n = (-\infty < X \leq -n)$ then $\{A_n\}$ is a contracting sequence of events such that $\lim_{n\to\infty} A_n = \varphi$

$\therefore P\left(\lim_{n\to\infty} A_n\right) = P(\varphi) = 0 ---(1)$

$P(A_n) = P(-\infty < X \leq -n) = F(-n) \therefore \lim_{n\to\infty} P(A_n) = \lim_{n\to\infty} F(-n) = F(-\infty) ---(2)$

$\because P\left(\lim_{n\to\infty} A_n\right) = \lim_{n\to\infty} P(A_n) \quad \therefore \; By\ (1)\ \&\ (2) \quad F(-\infty) = 0$

Let $A_n = (-\infty < X \leq n)$ then $\{A_n\}$ is a expanding sequence of events such that $\lim_{n\to\infty} A_n = S$

$\therefore P\left(\lim_{n\to\infty} A_n\right) = P(S) = 1 ---(1)$

$P(A_n) = F(n) \;\therefore \lim_{n\to\infty} P(A_n) = \lim_{n\to\infty} F(n) = F(\infty) ---(2)$

$\because P\left(\lim_{n\to\infty} A_n\right) = \lim_{n\to\infty} P(A_n) \quad \therefore \; By\ (1)\ \&\ (2) \quad F(\infty) = 1$

Let $A_n = \left(a - \frac{1}{n} < X \le a\right)$ then $\{A_n\}$ is a contracting sequence of events such that

$\lim_{n\to\infty} A_n = (X = a) \therefore P(\lim_{n\to\infty} A_n) = P(X = a) ---(1)$

$\text{P}(A_n) = F(a) - F\left(a - \frac{1}{n}\right) \therefore \lim_{n\to\infty} P(A_n) = F(a) - F(a - 0) ---(2)$

$\because P\left(\lim_{n\to\infty} A_n\right) = \lim_{n\to\infty} P(A_n) \quad \therefore \ By\ (1)\ \&\ (2) \quad F(a) - F(a - 0) = P(X = a)$

If $\text{P}(X = a) > 0$ then $F(x)$ has a jump discontinuity on the left at $x = a$

Let $A_n = \left(a < X \le a + \frac{1}{n}\right)$ then $\{A_n\}$ is a contracting sequence of events such that

$\lim_{n\to\infty} A_n = \varphi \therefore P(\lim_{n\to\infty} A_n) = 0 ---(1)$

$P(A_n) = F\left(a + \frac{1}{n}\right) - F(a) \therefore \lim_{n\to\infty} P(A_n) = F(a + 0) - F(a) ---(2)$

$\because P\left(\lim_{n\to\infty} A_n\right) = \lim_{n\to\infty} P(A_n) \quad \therefore \ By\ (1)\ \&\ (2)\ F(a + 0) = F(a)$

$\therefore F(x)$ is continuous on the right at all points

Ex: The spectrum of the random variable X consists of the points $1,2,---,n$ &

$P(X=i)$ is proportional to $\dfrac{1}{i(i+1)}$. determine the distribution function of X.

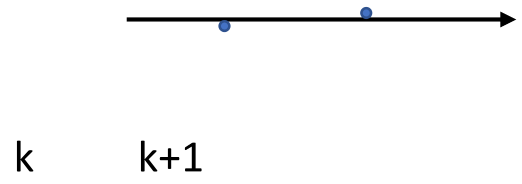
Compute $P(3<X\le n)$ & $P(X>5)$

Soln: Let $P(X=i)=\frac{k}{i(i+1)}$

$$\because 1=\sum_{i=1}^{n} P(X=i)=k\sum_{i=1}^{n}\frac{1}{i(i+1)}=k\sum_{i=1}^{n}\left(\frac{1}{i}-\frac{1}{i+1}\right)=k\left(1-\frac{1}{n}\right)=\frac{kn}{n+1}\ \therefore k=\frac{n+1}{n}$$

$$\therefore P(X=i)=\frac{n+1}{ni(i+1)}\quad If\ k\ is\ an\ integer\ then\ F(k)=P(X\le k)=\frac{n+1}{n}\sum_{i=1}^{k}\frac{i}{i+1}=\frac{k(n+1)}{n(k+1)}$$

$$If\ k\le x<k+1\ then\ F(x)=P(X\le x)=\sum_{i=1}^{k}P(X=i)=F(k)=\frac{k(n+1)}{n(k+1)}$$



$$P(3<X\le n)=F(n)-F(3)=1-\frac{3(n+1)}{4n}=\frac{n-3}{4n}$$

$$P(X>5)=1-P(X\le 5)=1-F(5)=1-\frac{5(n+1)}{6n}=\frac{n-5}{6n}$$

Ex: The distribution function $F(x)$ of a random variable X is defined as

$$F(x) = A;\ -\infty < x < -1$$

$$= B; -1 \leq x < 0$$

$$= C; 0 \leq x < 2$$

$= D; 2 \leq x < \infty$, where A, B, C & D are constants. Determine the values of A, B, C & D.

Given that $P(X = 0) = \frac{1}{6}$ & $P(X > 1) = \frac{2}{3}$

Soln: $A = F(-\infty) = 0,\ D = F(\infty) = 1,\ \frac{1}{6} = F(0) - F(0-0) = C - B$ --- (1)

$$\frac{2}{3} = P(X > 1) = 1 - P(X \leq 1) = 1 - F(1) = 1 - C \quad \therefore C = 1 - \frac{2}{3} = \frac{1}{3}$$

From (1) putting the value of C, $B = \frac{1}{3} - \frac{1}{6} = \frac{1}{6}$