BNC Mathematics Sem - IV (Fourier series)

Problem:

Represent a function f(x) by a trigonometrical series of the form

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n coxnx + b_n sinnx) ---(1)$$

Orthogonality formulae of integral:

1. $\int_{-\pi}^{\pi} sinnxdx = \int_{-\pi}^{\pi} sinmx\ cosnx\ dx = 0$ since sinnx & sinmx cosnx are odd function

2. $\int_{-\pi}^{\pi} cosnxdx = \begin{cases} 0, n > 0 \\ 2\pi, n = 0 \end{cases}$ since sinnπ = 0 for any integer n

3. $\int_{-\pi}^{\pi} sinmx\ sinnx\ dx = \begin{cases} 0, m \neq n \\ \pi, \quad m = n > 0 \\ 0, \quad m = n = 0 \end{cases}$

4. $\int_{-\pi}^{\pi} cosmx\ cosnx\ dx = \begin{cases} 0, m \neq n \\ \pi, m = n > 0 \\ 2\pi, m = n = 0 \end{cases}$

$where\ a_0, \{a_n\}, \{b_n\}\ are\ fourier\ coefficients$

Where $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx$$

And $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) sinnx\, dx$

We have to find these coefficients for a function f(x) which satisfies Dirichlet's conditions.

Now Dirichlet's conditions:

f(x)satisfies Dirichlet's conditions if f(x) is bounded, periodic, integrable in [-$\pi,\pi]$ *and piecewise monotone*. if f(x)=x^2 it is bounded, periodic, integrable and $f'(x) = 2x$ *which is* > 0 *in* $(0,\pi)$ *and* < 0 *in* $(-\pi, 0)$ $\therefore$ $f(x)$*is monotone increasing in* $(0,\pi)$ *and monotone decreasing in* $(-\pi,0)$ $\therefore$ $f(x)$*satisfies diriclet's conditions in* $[-\pi,\pi]$.*now find fourier coefficients*

Fourier series and Fourier constants:

Let f(x) can be represented by a series $\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n coxnx+b_n sinnx)---(1)$

and the series (1) converges uniformly to f(x) on $-\pi\leq x\leq\pi$

Then we have $f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n coxnx+b_n sinnx)---(2)$

Since the series (1) converges uniformly on $-\pi\leq x\leq\pi\ to\ f(x)$

$$\therefore \int_{-\pi}^{\pi} f(x)dx=\frac{a_0}{2}\int_{-\pi}^{\pi} dx+\sum_{n=1}^{\infty}(a_n\int_{-\pi}^{\pi} coxnx\,dx+b_n\int_{-\pi}^{\pi} sinnx\ \ dx)---(3)$$

$\therefore \int_{-\pi}^{\pi} f(x)dx=\frac{a_0}{2}\times 2\pi$ since $\int_{-\pi}^{\pi} coxnx\,dx=\int_{-\pi}^{\pi} sinnx\ \ dx=0\ by\ P1\ \&P2.$

$$\therefore a_0=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx$$

Multiply (2) by $coskx\ for\ integer\ k\geq 1\ it\ remains\ u-ly\ convergent\ in-\pi\leq x\leq\pi$

$$\int_{-\pi}^{\pi} f(x)coskxdx=\frac{a_0}{2}\int_{-\pi}^{\pi} coskxdx+\sum_{n=1}^{\infty}(a_n\int_{-\pi}^{\pi} coskxcoxnx\,dx+b_n\int_{-\pi}^{\pi} coskxsinnx\ \ dx)$$

$\therefore \int_{-\pi}^{\pi} f(x)coskxdx=\pi a_k\ by\ properties\ P1,P2,$ P4 $\therefore a_k=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)coskxdx$

Similarly multiplying (2) by $sinkx \ for \ integer \ k \geq 1$ $b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)sinkxdx$

series (1) is known as Fourier series corresponding to f(x) and is denoted by

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n coxnx + b_n sinnx) --- (2)$$

and $a_0, \{a_n\}, \{b_n\} are \ known \ as \ Fourier \ constants.$

Dirichlet's conditions:

A function $f(x) is \ said \ to \ satisfy \ Dirichlet's \ conditions \ on \ an \ interval$

$-\pi$ $\leq \pi$ in which it is defined when it subjected to one of the following cond.

1. $f(x) \ is \ bounded \ periodic \ with \ period \ 2\pi \ and \ integrable \ on -\pi \leq x \leq \pi$

And the val can be divided into a finite no. of open partial interval in each of which $f(x) is \ monotonic \ \{ \ or \ f(x) is \ bounded \ periodic \ with \ period \ 2\pi \ and$

Integrable on $-\pi \leq x \leq \pi$ and piecewise monotonic on $[-\pi, \pi]\}$

2. $f(x)$ *has a finite number of points of infinite discontinuties in the interval.* When arbitrary small neighbourhoods of these points are excluded, f(x) is bounded in the remainder of the interval, and this can be broken up into a finite number of open partial intervals in each of which f(x) is monotonic.

Convergence

When $f(x)$ *satisfies Dirichlet's conditions on* $-\pi \le x \le \pi$

The Fourier series corresponding to $f(x)$ *converges to* $f(x)$

at any point on $-\pi < x < \pi$ when $f(x)$ *is continuous.*

And converges to $\frac{1}{2}\{f(x+0)+f(x-0)\}$ *when there is a ordinary discontinuity at x*

In particular at $x = \pi$ *and at* $x = -\pi$ *it coverges to* $\frac{1}{2}\{f(-\pi+0)+f(\pi-0)\}$

when $f(-\pi+0)$ & $f(\pi-0)$ exists.

Where $f(\pi+0) = \lim_{x\to\pi+0} f(x)$ i.e right hand limit of $f(x)$ at $x=\pi$

Ex1: Verify that $f(x) = x^2$ *satisfies Dirichlet's conditions on* $[-\pi, \pi]$.
Show that the Fourier series corresponding to x^2 *on* $-\pi \leq x \leq \pi$ *is* $\frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}(-1)^n \frac{cosnx}{n^2}$. *Hence deduce that* $1 + \frac{1}{2^2} + \frac{1}{3^2} + --- = \frac{\pi^2}{6}$

& $1 - \frac{1}{2^2} + \frac{1}{3^2} - --- = \frac{\pi^2}{12}$

Soln.: $f(x) = x^2$ *is bounded* & integrable on $-\pi \leq x \leq \pi$, *since it is continuous on* $[-\pi, \pi]$
$f'(x) = 2x > 0$ *on* $0 < x \leq \pi$ *i.e.* $f(x)$ *is monotone increasing on* $0 < x \leq \pi$

< 0 *on* $-\pi \leq x < 0$ *i.e.* $f(x)$ *is motonic decreasing on* $-\pi \leq x < 0$

Therefore $f(x)$ *is piecewise monotone on* $[-\pi, \pi]$. Hence $f(x) = x^2$ satisfies Dirichlet's conditions on $[-\pi, \pi]$. Also $f(x) = x^2$ is an even function of x.

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{2}{\pi}\int_{0}^{\pi} x^2 dx = \frac{2\pi^2}{3}$$

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)coskxdx = \frac{2}{\pi}\int_{0}^{\pi} x^2 coskx\, dx = \frac{4}{k^2}cosk\pi = \frac{4(-1)^k}{k^2}$$ (integrating by parts)

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)sinkxdx = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 sinkx\, dx = 0$$ *since* $x^2 sinkx$ is an odd function of x.

Therefore Fourier series corresponding to $f(x) = x^2$ is $\frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}(-1)^n\frac{cosnx}{n^2}$.

i.e. $f(x) = x^2 \sim \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}(-1)^n\frac{cosnx}{n^2}$.

Since $f(x) satisfies\ Dirichlet's\ conditions\ and\ continuous\ on\ [-\pi, \pi]$

Therefore $f(x) = x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}(-1)^n\frac{cosnx}{n^2}\ on\ -\pi < x < \pi ---(1)$

at $x = 0\ from\ (1)\ 0 = \frac{\pi^2}{3} + 4\{-1 + \frac{1}{2^2} - \frac{1}{3^2} ---\} \Rightarrow 1 - \frac{1}{2^2} + \frac{1}{3^2} - ---= \frac{\pi^2}{12}$

$$\frac{1}{2}\{f(-\pi + 0) + f(\pi - 0)\} = \pi^2 = \frac{\pi^2}{3} + 4\left\{1 + \frac{1}{2^2} + \frac{1}{3^2} + ---\right\}$$

$$\therefore 1 + \frac{1}{2^2} + \frac{1}{3^2} + ---= \frac{\pi^2}{6}$$