Some Probability Inequalities

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1 Markov Inequality

Statement: If $P(X \geq 0) = 1$ and $E(X) < \infty$, then $P(X \geq t) \leq \frac{E(X)}{t}$ $\forall t > 0$.

Proof: Let X be a discrete random variable with p.m.f. 'p'. Then

$E(X) = \sum_{j:j\geq 0} jp(j) = \sum_{j:0\leq j<t} jp(j) + \sum_{j:j\geq t} jp(j) \geq \sum_{j:j\geq t} jp(j) \geq \sum_{j:j\geq t} tp(j) = t\sum_{j:j\geq t} p(j) = tP(X \geq t)$.

Hence the proof. Note that, inequalities have been used in two stages. So the equality holds iff the causes of inequality vanish in both stages, i.e. '=' holds iff $\{j : p(j) > 0 \text{ and } 0 \leq j < t\} = \phi$(Null Set) or $\{0\}$ and $\{j : p(j) > 0 \text{ and } j \geq t\} = \{t\}$. Combining both, '=' holds iff $\{j : p(j) > 0\} = \{t\}$ or $\{0, t\}$.

Implication: It provides a non-trivial upper bound (trivial upper bound is 1) in terms of a multiple of $E(X)$ to the right tail probability.

2 Chebyshev's Inequality

Statement: X is a random variable with $E(X) = \mu$ and $Var(X) = \sigma^2$. Then $P\{|X - \mu| < \sigma t\} \geq 1 - \frac{1}{t^2}$ $\forall t > 0$.

Proof: Let X be a discrete random variable having p.m.f. 'p'. Then

$\sigma^2 = E(X-\mu)^2 = \sum_{j:j-\mu\leq -\sigma t} (j-\mu)^2 p(j) + \sum_{j:-\sigma t<j-\mu<\sigma t} (j-\mu)^2 p(j) + \sum_{j:j-\mu\geq \sigma t} (j-\mu)^2 p(j) \geq \sum_{j:j-\mu\leq -\sigma t} (j-\mu)^2 p(j) + \sum_{j:j-\mu\geq \sigma t} (j-\mu)^2 p(j) = \sum_{j:|j-\mu|\geq \sigma t} (|j-\mu|)^2 p(j) \geq \sigma^2 t^2 \sum_{j:|j-\mu|\geq \sigma t} p(j) = P\{|X-\mu| \geq \sigma t\}$.

Hence, $P\{|X - \mu| < \sigma t\} \geq 1 - \frac{1}{t^2}$. '=' holds iff $\{j : p(j) > 0 \text{ and } \mu - \sigma t < j < \mu + \sigma t\} = \phi$ or μ and $\{j : p(j) > 0 \text{ and } |j - \mu| \geq \sigma t\} = \{\mu+\sigma t\}$ or $\{\mu-\sigma t\}$ or $\{\mu-\sigma t, \mu+\sigma t\}$. Combining both, '=' holds iff $\{j : p(j) > 0\} = \{\mu, \mu-\sigma t\}$ or $\{\mu, \mu+\sigma t\}$ or $\{\mu, \mu-\sigma t, \mu+\sigma t\}$.

Implication: It provides a non-trivial lower bound (trivial lower bound is 0) of the probability that the value of a random variable X falling in $(\mu - \sigma t, \mu + \sigma t)$.

3 Exercises

1. Derive Chebyshev's inequality using Markov's inequality.
2. $X \sim Poi(1)$, then find a non trivial upper bound of $P(X \geq 2)$ and non trivial lower bound of $P(X < 4)$.
3. If $X \sim Bin(n, p)$ then prove that $P\left(|\frac{X}{n} - p| \geq 0.5\right) \leq \frac{1}{n}$.
4. Practice relevant problems from Rohatgi and Saleh, Outline Vol-2 and other books on Probability.