$$\phi = U(x, y) + \lambda(M - P_x x - P_y . y)$$

F.O.C

$$\phi_x = U_x - \lambda P_x = 0 \quad \ldots\ldots (\text{I})$$

$$\phi_y = U_y - \lambda P_y = 0 \quad \ldots\ldots (\text{II})$$

$$\phi_\lambda = M - P_x . x - P_y . y \quad \ldots\ldots (\text{III})$$

$$\frac{U_x}{U_y} = \frac{P_x}{P_y} \quad \ldots\ldots \text{Equilibrium condition.}$$

$$x^* = x(P_x, P_y, M) \quad \ldots\ldots \text{demand function of } x$$

$$y^* = y(P_x, P_y, M) \quad \ldots\ldots \text{demand function of } y$$

$$\lambda^* = \lambda(P_x, P_y, M)$$

$$U^* = U(x^*, y^*) \quad \ldots\ldots \text{Indirect utility function.}$$

$$= U(P_x, P_y, M) \quad \ldots\ldots \text{Indirect utility function.}$$

$$U^* = U(x^*, y^*)$$

$$\frac{\partial U^*}{\partial M} = \frac{\partial U^*}{\partial x^*} \cdot \frac{\partial x^*}{\partial M} + \frac{\partial U^*}{\partial y^*} \cdot \frac{\partial y^*}{\partial M}$$

$$= U_x . \frac{\partial x^*}{\partial M} + U_2 \frac{d y^*}{\partial M}$$

$$= \lambda P_x . \frac{\partial x^*}{\partial M} + \lambda P_y . \frac{\partial y^*}{\partial M}$$

$$= \lambda \left[P_x . \frac{\partial x^*}{\partial M} + P_y . \frac{\partial y^*}{\partial M}\right]$$

$$M = P_x . x^* + P_y . y^*$$

Differentiating both side w.r.to M

$$1 = P_x . \frac{\partial x^*}{\partial M} + P_y . \frac{\partial y^*}{\partial M}$$

$$\therefore \frac{\partial U^*}{\partial M} = \lambda = \text{Marginal utility of money.}$$

S.O.C

$$\begin{vmatrix} U_{xx} & U_{xy} & -P_x \\ U_{yx} & U_{yy} & -P_y \\ -P_x & -P_y & 0 \end{vmatrix} > 0.$$